12.6 **Check Your Understanding** Consider using Ampère's law to calculate the magnetic fields of a finite straight wire and of a circular loop of wire. Why is it not useful for these calculations?

12.6 | Solenoids and Toroids

Learning Objectives

By the end of this section, you will be able to:

- Establish a relationship for how the magnetic field of a solenoid varies with distance and current by using both the Biot-Savart law and Ampère's law
- Establish a relationship for how the magnetic field of a toroid varies with distance and current by using Ampère's law

Two of the most common and useful electromagnetic devices are called solenoids and toroids. In one form or another, they are part of numerous instruments, both large and small. In this section, we examine the magnetic field typical of these devices.

Solenoids

A long wire wound in the form of a helical coil is known as a **solenoid**. Solenoids are commonly used in experimental research requiring magnetic fields. A solenoid is generally easy to wind, and near its center, its magnetic field is quite uniform and directly proportional to the current in the wire.

Figure 12.19 shows a solenoid consisting of N turns of wire tightly wound over a length L. A current I is flowing along the wire of the solenoid. The number of turns per unit length is N/L; therefore, the number of turns in an infinitesimal length dy are $(N/L)dy$ turns. This produces a current

$$dI = \frac{NI}{L}dy. \tag{12.24}$$

We first calculate the magnetic field at the point P of **Figure 12.19**. This point is on the central axis of the solenoid. We are basically cutting the solenoid into thin slices that are dy thick and treating each as a current loop. Thus, dI is the current through each slice. The magnetic field $d\vec{\mathbf{B}}$ due to the current dI in dy can be found with the help of **Equation 12.15** and **Equation 12.24**:

$$d\vec{\mathbf{B}} = \frac{\mu_0 R^2 dI}{2(y^2 + R^2)^{3/2}}\hat{\mathbf{j}} = \left(\frac{\mu_0 IR^2 N}{2L}\hat{\mathbf{j}}\right)\frac{dy}{(y^2 + R^2)^{3/2}} \tag{12.25}$$

where we used **Equation 12.24** to replace dI. The resultant field at P is found by integrating $d\vec{\mathbf{B}}$ along the entire length of the solenoid. It's easiest to evaluate this integral by changing the independent variable from y to θ. From inspection of **Figure 12.19**, we have:

$$\sin\theta = \frac{y}{\sqrt{y^2 + R^2}}. \tag{12.26}$$

(a)   (b)
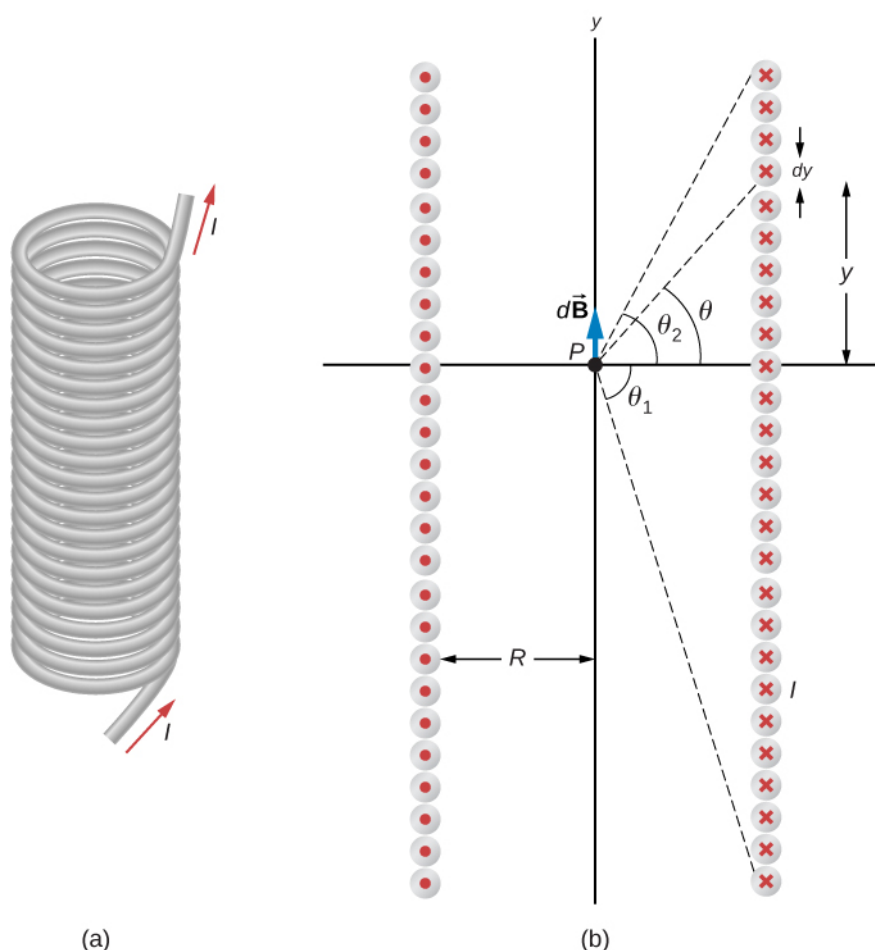
Figure 12.19 (a) A solenoid is a long wire wound in the shape of a helix. (b) The magnetic field at the point P on the axis of the solenoid is the net field due to all of the current loops.

Taking the differential of both sides of this equation, we obtain

$$\cos\theta \, d\theta = \left[-\frac{y^2}{(y^2 + R^2)^{3/2}} + \frac{1}{\sqrt{y^2 + R^2}} \right] dy$$

$$= \frac{R^2 \, dy}{(y^2 + R^2)^{3/2}}.$$

When this is substituted into the equation for $d\overrightarrow{\mathbf{B}}$, we have

(12.27)

$$\overrightarrow{\mathbf{B}} = \frac{\mu I_0 N}{2L} \hat{\mathbf{j}} \int_{\theta_1}^{\theta_2} \cos\theta \, d\theta = \frac{\mu I_0 N}{2L} (\sin\theta_2 - \sin\theta_1) \hat{\mathbf{j}},$$

which is the magnetic field along the central axis of a finite solenoid.

Of special interest is the infinitely long solenoid, for which $L \to \infty$. From a practical point of view, the infinite solenoid is one whose length is much larger than its radius $(L \gg R)$. In this case, $\theta_1 = \frac{-\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$. Then from **Equation 12.27**, the magnetic field along the central axis of an infinite solenoid is

$$\overrightarrow{\mathbf{B}} = \frac{\mu_0 I N}{2L} \hat{\mathbf{j}} [\sin(\pi/2) - \sin(-\pi/2)] = \frac{\mu_0 I N}{L} \hat{\mathbf{j}}$$

or

$$\overrightarrow{\mathbf{B}} = \mu_0 nI \,\hat{\mathbf{j}},$$ (12.28)

where n is the number of turns per unit length. You can find the direction of $\overrightarrow{\mathbf{B}}$ with a right-hand rule: Curl your fingers in the direction of the current, and your thumb points along the magnetic field in the interior of the solenoid.

We now use these properties, along with Ampère's law, to calculate the magnitude of the magnetic field at any location inside the infinite solenoid. Consider the closed path of **Figure 12.20**. Along segment 1, $\overrightarrow{\mathbf{B}}$ is uniform and parallel to the path. Along segments 2 and 4, $\overrightarrow{\mathbf{B}}$ is perpendicular to part of the path and vanishes over the rest of it. Therefore, segments 2 and 4 do not contribute to the line integral in Ampère's law. Along segment 3, $\overrightarrow{\mathbf{B}} = 0$ because the magnetic field is zero outside the solenoid. If you consider an Ampère's law loop outside of the solenoid, the current flows in opposite directions on different segments of wire. Therefore, there is no enclosed current and no magnetic field according to Ampère's law. Thus, there is no contribution to the line integral from segment 3. As a result, we find

$$\oint \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{l}} = \int_1 \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{l}} = Bl.$$ (12.29)
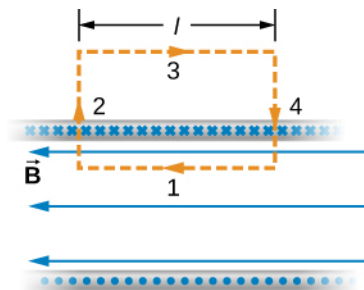
Figure 12.20 The path of integration used in Ampère's law to evaluate the magnetic field of an infinite solenoid.

The solenoid has n turns per unit length, so the current that passes through the surface enclosed by the path is nlI. Therefore, from Ampère's law,

$$Bl = \mu_0 nlI$$

and

$$B = \mu_0 nI$$ (12.30)

within the solenoid. This agrees with what we found earlier for B on the central axis of the solenoid. Here, however, the location of segment 1 is arbitrary, so we have found that this equation gives the magnetic field everywhere inside the infinite solenoid.

Outside the solenoid, one can draw an Ampère's law loop around the entire solenoid. This would enclose current flowing in both directions. Therefore, the net current inside the loop is zero. According to Ampère's law, if the net current is zero, the magnetic field must be zero. Therefore, for locations outside of the solenoid's radius, the magnetic field is zero.

When a patient undergoes a magnetic resonance imaging (MRI) scan, the person lies down on a table that is moved into the center of a large solenoid that can generate very large magnetic fields. The solenoid is capable of these high fields from high currents flowing through superconducting wires. The large magnetic field is used to change the spin of protons in the patient's body. The time it takes for the spins to align or relax (return to original orientation) is a signature of different tissues that can be analyzed to see if the structures of the tissues is normal (**Figure 12.21**).

Figure 12.21 In an MRI machine, a large magnetic field is generated by the cylindrical solenoid surrounding the patient. (credit: Liz West)

Example 12.9

Magnetic Field Inside a Solenoid

A solenoid has 300 turns wound around a cylinder of diameter 1.20 cm and length 14.0 cm. If the current through the coils is 0.410 A, what is the magnitude of the magnetic field inside and near the middle of the solenoid?

Strategy

We are given the number of turns and the length of the solenoid so we can find the number of turns per unit length. Therefore, the magnetic field inside and near the middle of the solenoid is given by **Equation 12.30**. Outside the solenoid, the magnetic field is zero.

Solution

The number of turns per unit length is

$$n = \frac{300 \text{ turns}}{0.140 \text{ m}} = 2.14 \times 10^3 \text{ turns/m.}$$

The magnetic field produced inside the solenoid is

$$B = \mu_0 nI = (4\pi \times 10^{-7} \text{ T} \cdot \text{m/A})(2.14 \times 10^3 \text{ turns/m})(0.410 \text{ A})$$
$$B = 1.10 \times 10^{-3} \text{ T.}$$

Significance

This solution is valid only if the length of the solenoid is reasonably large compared with its diameter. This example is a case where this is valid.

12.7 Check Your Understanding What is the ratio of the magnetic field produced from using a finite formula over the infinite approximation for an angle θ of (a) 85°? (b) 89°? The solenoid has 1000 turns in 50 cm with a current of 1.0 A flowing through the coils

Toroids

A toroid is a donut-shaped coil closely wound with one continuous wire, as illustrated in part (a) of **Figure 12.22**. If the toroid has N windings and the current in the wire is I, what is the magnetic field both inside and outside the toroid?
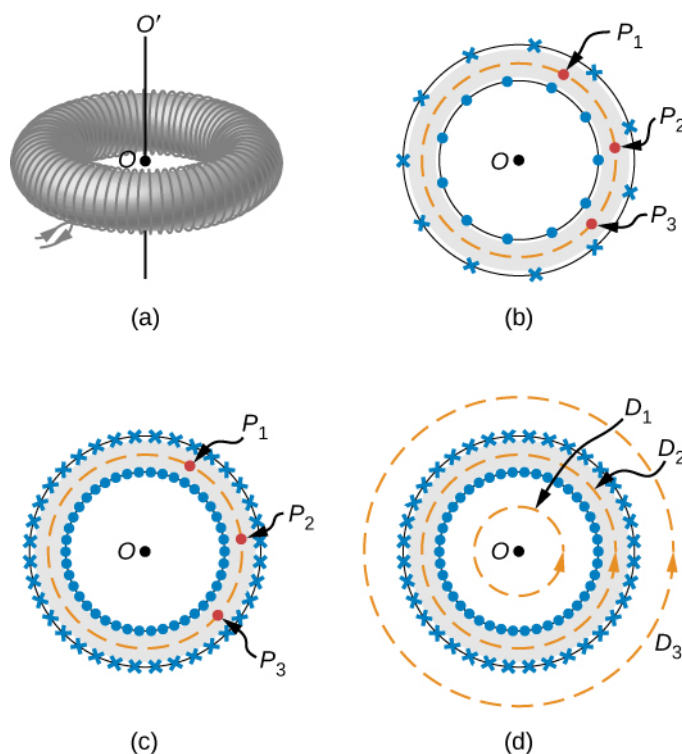
Figure 12.22 (a) A toroid is a coil wound into a donut-shaped object. (b) A loosely wound toroid does not have cylindrical symmetry. (c) In a tightly wound toroid, cylindrical symmetry is a very good approximation. (d) Several paths of integration for Ampère's law.

We begin by assuming cylindrical symmetry around the axis OO'. Actually, this assumption is not precisely correct, for as part (b) of **Figure 12.22** shows, the view of the toroidal coil varies from point to point (for example, P_1, P_2, and P_3) on a circular path centered around OO'. However, if the toroid is tightly wound, all points on the circle become essentially equivalent [part (c) of **Figure 12.22**], and cylindrical symmetry is an accurate approximation.

With this symmetry, the magnetic field must be tangent to and constant in magnitude along any circular path centered on OO'. This allows us to write for each of the paths D_1, D_2, and D_3 shown in part (d) of **Figure 12.22**,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = B(2\pi r). \tag{12.31}$$

Ampère's law relates this integral to the net current passing through any surface bounded by the path of integration. For a path that is external to the toroid, either no current passes through the enclosing surface (path D_1), or the current passing through the surface in one direction is exactly balanced by the current passing through it in the opposite direction (path D_3). In either case, there is no net current passing through the surface, so

$$\oint B(2\pi r) = 0$$

and

$$B = 0 \quad \text{(outside the toroid).} \tag{12.32}$$

The turns of a toroid form a helix, rather than circular loops. As a result, there is a small field external to the coil; however, the derivation above holds if the coils were circular.

For a circular path within the toroid (path D_2), the current in the wire cuts the surface N times, resulting in a net current NI through the surface. We now find with Ampère's law,

$$B(2\pi r) = \mu_0 NI$$

and

$$B = \frac{\mu_0 NI}{2\pi r} \quad \text{(within the toroid).}$$

<div align="right">(12.33)</div>

The magnetic field is directed in the counterclockwise direction for the windings shown. When the current in the coils is reversed, the direction of the magnetic field also reverses.

The magnetic field inside a toroid is not uniform, as it varies inversely with the distance r from the axis OO'. However, if the central radius R (the radius midway between the inner and outer radii of the toroid) is much larger than the cross-sectional diameter of the coils r, the variation is fairly small, and the magnitude of the magnetic field may be calculated by **Equation 12.33** where $r = R$.

12.7 | Magnetism in Matter

Learning Objectives

By the end of this section, you will be able to:

- Classify magnetic materials as paramagnetic, diamagnetic, or ferromagnetic, based on their response to a magnetic field
- Sketch how magnetic dipoles align with the magnetic field in each type of substance
- Define hysteresis and magnetic susceptibility, which determines the type of magnetic material

Why are certain materials magnetic and others not? And why do certain substances become magnetized by a field, whereas others are unaffected? To answer such questions, we need an understanding of magnetism on a microscopic level.

Within an atom, every electron travels in an orbit and spins on an internal axis. Both types of motion produce current loops and therefore magnetic dipoles. For a particular atom, the net magnetic dipole moment is the vector sum of the magnetic dipole moments. Values of μ for several types of atoms are given in **Table 12.1**. Notice that some atoms have a zero net dipole moment and that the magnitudes of the nonvanishing moments are typically $10^{-23}\,\text{A} \cdot \text{m}^2$.

Atom	Magnetic Moment $\left(10^{-24}\,\text{A} \cdot \text{m}^2\right)$
H	9.27
He	0
Li	9.27
O	13.9
Na	9.27
S	13.9

Table 12.1 Magnetic Moments of Some Atoms

A handful of matter has approximately 10^{26} atoms and ions, each with its magnetic dipole moment. If no external magnetic field is present, the magnetic dipoles are randomly oriented—as many are pointed up as down, as many are pointed east as west, and so on. Consequently, the net magnetic dipole moment of the sample is zero. However, if the sample is placed in a magnetic field, these dipoles tend to align with the field (see **Equation 12.14**), and this alignment determines how the sample responds to the field. On the basis of this response, a material is said to be either paramagnetic, ferromagnetic, or diamagnetic.

In a **paramagnetic material**, only a small fraction (roughly one-third) of the magnetic dipoles are aligned with the applied field. Since each dipole produces its own magnetic field, this alignment contributes an extra magnetic field, which enhances the applied field. When a **ferromagnetic material** is placed in a magnetic field, its magnetic dipoles also become aligned;